# Residually Finite Groups and Hyperbolicity

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November 21, 2022

## Motivation.

Although there are plenty of exotic finite groups, Geometric Group Theory has focused primarily on infinite groups. However, finite groups have some very convenient properties. It is therefore valuable to consider infinite groups with properties that are similar to those of finite groups. As it stands, this post is principally inspired by [2]. We'll begin by answering what it means to be residual.

**Definition 1.** *For a given property $P$, a group $G$ is said to be* ***residually*** *$P$ if for any $g \neq e \in G$, there exists a normal subgroup $K \trianglelefteq G$ such that $g \notin K$ and $G/K$ has property $P$.*

Although questions about residual properties can motivate many different areas of study, we will focus on the following example:

## Residual Finitude.

**Definition 2.** *A group $G$ is said to be* ***residually finite*** *if for every $g \neq e \in G$, there exists a normal subgroup $K \trianglelefteq G$ such that $K$ has finite index with respect to $G$ and $g \notin K$.*

It follows from this definition that all finite groups are residually finite, since $\langle e \rangle$ is normal to all groups, and always has finite index to finite groups. Our principal interest here is going to be questions about the residual finitude of infinite groups. GGT has had us playing around with infinite groups and their corresponding symmetry groups, also known as automorphism groups:

**Definition 3.** *The* ***symmetry group*** *(also known as the* ***automorphism group****) of a group $G$ is the group formed by the set of automorphisms on $G$ endowed with the operation of composition.*

As it happens, we can obtain something nice involving symmetry groups and residual finitude.

**Theorem 1.** *The symmetry group of a finitely generated residually finite group $G$ is itself residually finite.*

To prove this theorem, we'll need some machinery. First, we'll need a lemma, which requires the following definition.

**Definition 4.** *Let $G$, $H$ be groups with $H \leq G$. If every automorphism $f$ of $G$ satisfies $f(H) = H$, then $H$ is a* ***characteristic*** *subgroup of $G$.*

**Example.** Let $G = \mathbb{Z}_4 = \langle a \mid a^4 \rangle$. Then the automorphisms of $G$ are the identity automorphism $e$ and the automorphism $g : a \mapsto a^3$. We observe that reduced elements of the form 0 and $a^2$ are invariant under both $e$ and $g$, so the subgroup $\langle a^2 \mid a^4 \rangle \leq G$ is *characteristic.*

**Lemma 1.** *For groups $G, H$ with $H \leq G$, let $H$ be characteristic to $G$. Then there is a homomorphism given by*

$$\psi : Sym(G) \to Sym(G/H)$$

*where for $\varphi \in Sym(G)$, $\psi(\varphi) = \varphi(G/H)$.*

**Proof.** It follows directly that $\psi(e) = e$. Let $\varphi, \varphi' \in \text{Sym}(G)$ be non-trivial automorphisms of $G$. Then $\psi(\varphi\varphi') = \varphi(G/H)\varphi'(G/H) = \psi(\varphi)\psi(\varphi')$.

Equipped with some facts about characteristic subgroups, we are ready to prove Theorem 1.

**Proof.** Let $\Phi = \text{Sym}(G)$ be the symmetry group of $G$, and let $\varphi \in \Phi$ such that $\varphi \neq e$. Since $\varphi$ isn't the identity automorphism, there must be at least one $g \in G$ such that

$$h = \varphi(g)g^{-1} \neq e.$$

Since $G$ is residually finite by assumption, there is some $K$ normal to $G$ with $[G : K] = n \in \mathbb{N}$ such that $h \notin K$. Further define $K^*$ as the intersection of all subgroups of $G$ which are normal to $G$ with index $n$. $K^*$ itself must have finite index with respect to $G$, and $K^*$ must be characteristic with respect to $G$ [1].

We now want to say something about $\text{Sym}(G/K^*) = \Phi'$. Since $K^*$ is characteristic, we know that the map

$$\psi : \Phi \to \Phi'$$

is a homomorphism by Lemma 1. Fortuitously, $\psi(\varphi) = \varphi' \neq e$, since $\varphi$ permutes $h \notin K^*$, so $h \in G/K^*$ and thus $\varphi'$ must be non-trivial.

Importantly $[\Phi : \Phi'] = 1$ and $\ker \psi = e$, which follows from the fact that $K^*$ is characteristic and thus invariant under automorphism. Since our choice of $\varphi$ was arbitrary, residual finitude implies the existence of a $K^*$ for any $\varphi$, and the subgroup of automorphisms $\Phi'$ associated with $G/K^*$ has finite index to $\Phi$, it must be the case for finitely-generated residually finite groups that their symmetry group $\Phi$ is residually finite. □

Not only is this a tasty result, it's due to Gilbert Baumslag, one of the people who (didn't) discover Baumslag-Solitar groups!

**Corollary 1.** *Free groups are residually finite.*

To prove this corollary, we'll make an argument about matrices, and to do so we'll want a lemma about free abelian groups. Recall that free abelian groups are, as the name suggests, like free groups but abelian; they can be thought of as an iterated direct product.

**Lemma 2.** *Free abelian groups are residually finite.*

**Proof.** Let $G = \mathbb{Z}^n = \mathbb{Z} \oplus \mathbb{Z} \oplus \cdots \oplus \mathbb{Z}$ be a free abelian group. Since $G$ is abelian, every subgroup of $G$ is normal. Let $h = (a, b, \ldots, n)$ for $a \neq e$. The subgroup $K = \mathbb{Z}/a\mathbb{Z} \oplus \mathbb{Z} \oplus \cdots \oplus \mathbb{Z}$ has index $a$ with respect to $G$, and $h \notin K$. Since our choice of $h$ was arbitrary, the lemma holds, and we're ready to prove Corollary 1.

**Proof.** From our demonstration of the Ping Pong Lemma a few weeks ago, we know

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$$

defined by matrix multiplication suffices to generate the free group $F_2$. But we also know that these matrices are automorphisms of the free abelian group of rank two. Specifically, the free abelian of rank two can be given as $G = \langle x, y \mid xy = yx \rangle$ over addition, and for $a \in G$ we have $Aa = (x + 2y, y)$ and $Ba = (x, 2x + y)$. A bit of computation confirms these are automorphisms.

We'll quickly detour to point out that every subgroup $H \leq \text{Sym}(G)$ is residually finite, which follows from a bit of casework. Detour in hand, we invoke Lemma 2, namely that free abelians groups are residually finite; by Theorem 1 the symmetry group of the free abelian of rank 2 must be residually finite, which implies that $F_2$ is a subgroup of a residually finite group. Thus $F_2$ must be residually finite.

We showed in class that for all $n \in \mathbb{N}$ we have $F_n \leq F_2$, and so we conclude that since $F_2$ is residually finite, all free groups are residually finite. □

We know free groups are pretty friendly, and now we have that free groups are residually finite, which helps us believe that residual finitude is a friendly thing for a group to be. We'll emphasize that even more by tying residual finitude to being Hopfean.

# Residual Finitude and Being Hopfian

**Recall 1.** *A group $G$ is **Hopfian** if there is no non-trivial quotient group $G/K$ such that $G \cong G/K$.*

**Theorem 2.** *Let $G$ be a finitely generated residually finite group. Then $G$ is Hopfian.*

To prove this works, we'll need two more lemmas:

**Lemma 3.** *Let $G$ be a finitely generated group. Every normal subgroup $K \trianglelefteq G$ of finite index contains a normal subgroup of finite index with respect to $G$.*

**Proof.** This proof comes from [3]. Reproducing the proof in its entirety would require three additional lemmas, and so it is omitted here for brevity.

**Lemma 4.** *Let $W \subseteq \mathcal{L}$ be a set of words on the language $\mathcal{L}$. If groups $G_1$ and $G_2$ are isomorphic, then $\Gamma_1 \cong \Gamma_2$, where*

$$\Gamma_1 = G_1/G_1(W).$$
$$\Gamma_2 = G_2/G_2(W)$$

In other words, adding the action of quotienting is similar to adding relators and should have the same impact on isomorphic groups.

**Proof.** We may let $\varphi$ be a group homomorphism from $G_1$ to $G_2$. This means that $G_1(W, \ldots)$ can be mapped injectively onto $G_2(W, \ldots)$, and since $G_1 \cong G_2$, we can also map the cosets of $\Gamma_1$ isomorphically to the cosets of $\Gamma_2$.

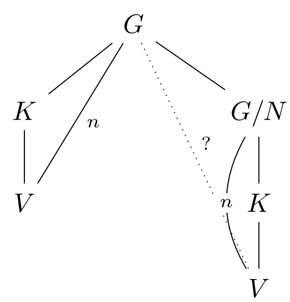
We're now ready to prove Theorem 2.

**Proof.** Towards a contradiction, let $N \trianglelefteq G$ such that $N \neq 1$ satisfies $G/N := G' \cong G$. We'll seek to show that this fails to work because mapping $G \to G'$ will to map non-trivial elements to $e$.

Consider some $g \neq 1 \in G$. By assumption of residual finitude, there is some finite index $K \trianglelefteq G$ such that $g \notin K$. By Lemma 3, there is a group $V \leq K$ such that $[G : V]$ is finite, and by assumption $V$ should have the same index to $G'$. We're now ready to invoke Lemma 4: if $G \cong G'$, then $G/V \cong G'/V$. Consider the canonical map

$$\psi : G/V \to G'/V.$$

We anticipate these quotients to be equivalent. However, we observe from the following subgroup lattice that they must be different:




This gives us our desired contradiction. □

Theorem 2 had as a condition that $G$ be residually finite. Does the result hold if $G$ isn't residually finite? No!

**Counterexample.** Baumslag-Solitar groups are finitely-generated and (mostly) non-Hopfian, as per Homework 3. It's non-constructive, but we can safely conclude that Baumslag-Solitar groups are not residually finite.

# Some Conclusions.

Every known hyperbolic group is residually finite. However, it remains an open question whether it is necessecary for hyperbolic groups to be residually finite. Like hyperbolicity, residual finitude is a condition that endowed to "nice" groups; the property of being Hopfian is a good example of this "nice" behavior.

# Works Cited.

[1] Hall, M. (1949). Coset representations in free groups. *Transactions of the American Mathematical Society, 67(2), 421–432.*

[2] Magnus, W., Karrass, A., & Solitar, D. (1976). *Combinatorial Group Theory*, 413-417. Dover Publications.

[3] Neumann, H. (1967). *Varieties of Groups.* Springer-Verlag.