Geometric Group Theory Final Project
Fuchsian Groups

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November 2022

1 Introduction

I aim to provide a short survey on Fuchsian groups. I'll start off by talking about the hyperbolic metric on the upper half plane, then talk about the projective analog of the special linear group and how it acts on the upper half plane by isometries. Then, I'll give a brief description of topological groups and their discrete subgroups. We then have enough machinery to define Fuchsian groups! The paper ends with a proof of the theorem that every abelian Fuchsian group is cyclic.

2 Hyperbolic Geometry and $\mathrm{PSL}_2(\mathbb{R})$

We can think of the hyperbolic metric loosely as the Euclidean distance over the y coordinate. It is written out more rigorously below.

$$ds = \frac{\sqrt{dx^2+dy^2}}{y}$$

This metric leads to very interesting things in its own right. Geodesics in this space are centered semi-circles and verticle lines. There is even a notion of a 'biggest triangle' under this metric. Additionally, no squares (four sided geodesic quadrilaterals) can exist under this metric. From now on, I'll notate $\mathbb{H}$ to be the upper half plane equipped with the hyperbolic metric.

A tiling of the hyperbolic upper half plane. Each figure is congruent under the hyperbolic metric

We now shift gears a little bit to talk about *Möbius (or fractional linear) transformations*. Möbius transformations are transformations of the complex plane, given by the form below.

$$\{z \to \tfrac{az+b}{cz+d} | a, b, c, d \in \mathbb{R}\}$$

It actually turns out that we can view these fractional linear transformations as matrices, where composing the transformations corresponds to matrix multiplication! This is not at all obvious at first, so bear with the calculations for a second.

$$\text{Let } \alpha = \frac{az+b}{cz+d} \text{ and } \beta = \frac{ez+f}{gz+h}$$

We compose $\alpha \cdot \beta$

$$= \frac{a(\frac{ez+f}{gz+h}) + b}{c(\frac{ez+f}{gz+h}) + d}$$
$$= \frac{a(ez+f) + b(gz+h)}{c(ez+f) + d(gz+h)}$$
$$= \frac{(ae+bg)z + (af+bh)}{(ce+dg)z + (cf+dh)}$$

Notice now if we think α and β as matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $\begin{pmatrix} e & f \\ g & h \end{pmatrix}$ respectively, the product $\alpha\beta$ is $\begin{pmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{pmatrix}$, precisely the coefficients in our composed fractional linear transformation.

I now state a pretty important theorem without a complete proof (the proof is very computationally heavy and I don't think the proof itself adds much value to this report.

Theorem: $\mathrm{PSL}_2((\mathbb{R}) \subset \mathrm{Isom}(\mathbb{H})$

What is this mystery group named $\mathrm{PSL}_2(\mathbb{R})$? It's simply $\mathrm{SL}_2(\mathbb{R})/\pm 1$, where 1 represents the two by two identity matrix. We have to mod out by this because we are using the upper half plane model for the hyperbolic plane, and we'd like isometries of the upper half plane to be sent to the upper half plane. The proof involves showing that elements of $\mathrm{PSL}_2(\mathbb{R})$ maps geodesics to geodesics, and is thus an isometry. With this connection between fractional linear transformations and viewing it as a more familiar matrix group, we are almost ready to define Fuchsian groups!

3 Topological Groups and Discrete Subgroups

An important part of the definition of a Fuchsian group is that it is a *topological group*. Loosely speaking, a topological group is a topological space that is also a group. More specifically, we require both the group operation and taking inverses to be continuous. This means that an equivalence in the sense of topological groups is both a group isomorphism and a homeomorphism of the underlying topological spaces.

Making sure these operations are continuous is often the trickiest part of defining a topological group. However, a set where the topology is very nicely behaved makes this very easy. This is precisely the idae behind a *discrete subgroup*. A discrete subgroup is a subgroup of a topological group that inherits the discrete topology. We can think about this inheritance as ignoring whatever topological structure was put on the parent group. More specifically, the discrete defines every element of the space to be open. If you are curious about the rigorous definitions, I'd urge you to look at the book *Fuchsian Groups* by Svetlana Katok. The value of discrete subgroups comes when we look at actions of the group: the action is properly discontinuous.

In the case of $\mathrm{PSL}_2(\mathbb{R})$, it is not immediately clear where it gets its topology from. It is a long string of inheritances: since it is a matrix group, it inherits our familiar Euclidean topology from $\mathbb{R}^4$, and we then identify $(a, b, c, d) \mapsto (-a, -b, -c, -d)$.

4 Fuchsian Groups

We now define a **Fuchsian group** as a discrete subgroup of $\mathrm{PSL}_2(\mathbb{R})$, or equivalently as a group that acts properly discontinuously on $\mathbb{H}$.

One example we've already seen is simply $\mathrm{PSL}_2(\mathbb{Z})$, which is also known as the modular group. It follows from the definition that any subgroup of a Fuchsian group is also Fuchsian.

We'd like to characterize the elements in any Fuchsian group, and to do so, we look to its parent group $\mathrm{PSL}_2(\mathbb{R})$. One way that may become clear after reminding ourselves that the group is made up of fractional linear transformations is to look at the fixed points of any given element.

Recall, fractional linear transformations are of the form

$$\{z \to \tfrac{az+b}{cz+d} | a, b, c, d \in \mathbb{R}, ad - bc = 1\}$$

To find fixed points, we set them equal

$$z = \tfrac{az+b}{cz+d}.$$

With some algebra, we get

$$cz^2 - (d-a)z - b = 0.$$

From here, we consider two cases, $c \neq 0$ and $c = 0$. If $c \neq 0$, we can use the quadratic formula to solve to get

$$z = \tfrac{a-d\pm\sqrt{(d-a)^2+4bc}}{2c}.$$

The important part of this equation is the discriminant as it tells us where our fixed points will be! We solve

$$\begin{aligned}(d-a)^2 + 4bc &= d^2 + a^2 - 2ad + 4bc \\ &= d^2 + a^2 - 2ad + 4(ad-1) \\ &= d^2 + a^2 + 2ad - 4 \\ &= (d+a)^2 - 4\end{aligned}$$

So it depends only on a and d, which is our trace of our matrix! Since we're modding out by ± 1, let's define our Trace function Tr as $|a + d|$. If $Tr(T) < 2$, then the fixed points are complex and conjugate to each other, so exactly one will be in $\mathbb{H}$. If $Tr(T) = 2$, then one point will be in $\mathbb{R}$, and if $Tr(T) > 2$, then both points are fixed in $\mathbb{R}$

In the case that $c = 0$, we get $ad = 1$, and our equation for finding fixed points becomes $z = a^2z - ba$. Note that ∞ is a possible solution for this equation! By plugging in values similar to the method above, we are able to classify elements in $\mathrm{PSL}_2(\mathbb{R})$ purely based on this trace function.

- If $Tr(T) > 2$ we call T *hyperbolic*
- If $Tr(T) = 2$ we call T *parabolic*
- If $Tr(T) < 2$ we call T *elliptic*

Recall from linear algebra that our trace function is basis-independent! In the context of a group, this means that trace is also preserved under conjugation by group elements. This lets us have a sort of canonical matrix form for each type of element that will prove to be very useful in proving our following claim.

Proposition 1 *Every Fuchsian group is abelian if and only if it is cyclic*

To prove this, we first prove several lemmas.

Lemma 1 *Given $T, S \in \mathrm{PSL}_2(\mathbb{R})$, if S and T commute, then they must have the same number of fixed points.*

Proof: If $Tp = p$, then $STp = Sp$. Since $TS = ST$, we get that $STp = TSp = Sp$, or, that T also fixes Sp.

Lemma 2 *Two nontrivial elements of $\mathrm{PSL}_2(\mathbb{R})$ commute if and only if they have the same fixed point set.*

Proof: We'll proceed by a case analysis on the three types of elements.

In the case of one fixed point, we have two elements T, S that commute. Since they commute, T maps the fixed point of S to itself injectively (follows from the lemma above). Furthermore, S also maps the fixed point of T to itself. So they fix each other's fixed points, and since this is the case where there is only one fixed point, that point must be the same point.

In the case that our elements are hyperbolic (they have two fixed points), we can use conjugates to put our matrix T into a form

$$\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$$

Where $\lambda \neq 1$. This element fixes the points 0 and ∞. We don't know about the fixed points of S, just that it commutes with T, so we write

$$\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a\lambda & b\lambda^{-1} \\ c\lambda & d\lambda^{-1} \end{pmatrix}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} = \begin{pmatrix} a\lambda & b\lambda \\ c\lambda^{-1} & d\lambda^{-1} \end{pmatrix}$$

These are only ever equal in the cases that $\lambda = 1$, which is not possible since our element is hyperbolic (if our element was elliptic or parabolic then see the case of one fixed point above), or the case that $b = c = 0$, which gives us that S also fixes the points 0 and ∞.

We now prove that two elements having the same fixed point set implies that they commute.

If we are given two elements that have the same fixed point set, which means that these elements are of the same type! This means we can use the *same* conjugator to send them to a form that commutes with itself. Since an element of this form commutes with another element of the same form, then the conjugates (when mapped by the same conjugator) will also commute.

We can conjugate an elliptic element to have the form

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

Similarly, for a parabolic element we get the form.

$$\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$$

And for a hyperbolic element, we get our old friend

$$\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$$

Every elememt of each type commutes with any other element in its own type! Since our conjugates commute, the elements also commute.

We are now well - armed enough to prove our original claim.

Theorem 3 *Every Fuchsian group is abelian if and only if it is cyclic*

Proof: We know that any cyclic group is abelian, so we immediately jump into proving the other direction.

If we have an abelian Fuchsian group, all of its elements must commute with each other. From Lemma 2, we get that they fix the same points. Finally, we know that elements that have the same fixed point set, must be of the same type, so we get that all nontrivial elements are of the same type. This means that we can conjugate every single element by the same conjugator to send the elements to our nicer forms.

For hyperbolic elements, we have the matrix

$$\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$$

We can map these elements isomorphically to a discrete subgroup of $(\mathbb{R}*, \cdot)$ under the map λ^2, which is cyclic!

For parabolic elements we have the conjugated matrix

$$\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$$

We can map these elements isomorphically to a discrete subgroup of $(\mathbb{R}, +)$ under the map x, which is cyclic!

Finally, we have elliptic elements we get

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

We map this to a discrete subgroup of $S^1, \cdot_{\mathbb{C}}$ under the function $e^{i\theta}$. Again, this is a cyclic group!

We have shown in every case, that our abelian group is isomorphic to a cyclic group!

This has neat corollary, namely that no Fuchsian group is isomorphic to $\mathbb{Z} \times \mathbb{Z}$, because it is an abelian but not cyclic group.

5 References

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