Braid Groups

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MATH 395

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1 Braid Groups

The following will be a brief introduction to braid groups which are groups that are presented as braids... But they are a lot more than that! After a brief introduction to mathematical *braids* and some neat examples, we will discuss the *pure braid group*. Afterward, we will discuss how braid groups relate to the symmetric group, a fundamental group, and some big theorems that relate to what we have accomplished in the course. Without further ado, here is the definition of a geometric braid:

Definition 1.1. A geometric braid is a geometric subset $\beta \subset \mathbb{R}^2 \times [0,1]$ such that β is composed of n disjoint topological intervals. Additionally, β must satisfy the following conditions:

$$\beta \cap (\mathbb{R}^2 \times 0) = \{(1,0,0),(2,0,0),\dots(n,0,0)\}$$
$$\beta \cap (\mathbb{R}^2 \times 1) = \{(1,0,1),(2,0,1),\dots(n,0,1)\}$$
$$\beta \cap (\mathbb{R}^2 \times t) \text{ consists of n points for all } t \in [0,1].$$

In addition, for any string in β, there exists a projection $proj_i : \mathbb{R}^2 \times [0,1] \mapsto [0,1]$. This takes that string homeomorphically to the unit interval.

The notation here might be a bit confusing at first, but the Cartesian product of $\mathbb{R}^2$ with $\{0\}$, $\{1\}$, and $\{t\}$ displays the permutation of the 0 endpoints, relative to the 1 endpoints. This can be further understood in Fig. 1 as the left endpoints being permuted with respect to the right ones. As is usual, some pictures of braids are bound to help us visualize and better understand these concepts. Pictured in Fig. 1 is a 3-string braid with two crossings and a 'usual' 3-string braid pattern in Fig. 2. To relate the figures to the definition above, we can think of each string within the brand traveling through all real numbers from 0 to 1, while the ends are fixed at some values within the set $[0,1]$. The conditions imposed on the braid are seen in these pictures as well, however, a more thorough explanation should help allow us to make a group structure out of these cool designs. First, we need to explain how to compose braids in a braid group. Intuitively, this makes sense since we can just stick them together at the ends. This relies on the first two conditions, which make the ends of the braids composable. Furthermore, strings are not allowed to intersect each other, but rather one must go above the other. This is shown in Fig. 1 where the top strings have no break throughout while the underlying strings have a gap around the intersection. The third

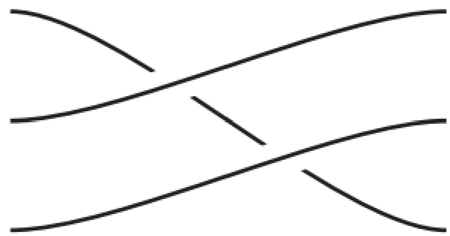

Figure 1: A 3-string braid with two crossings.

Figure 2: A ‘usual’ 3-string braid pattern.

Figure 3: A violation of the third condition from Definition 1.1 where the top strand loops over itself.

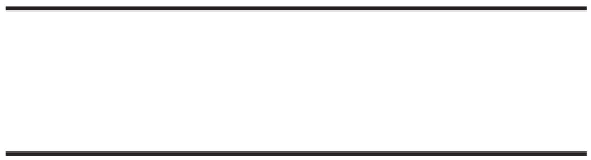

Figure 4: Indentity braid in B_2.

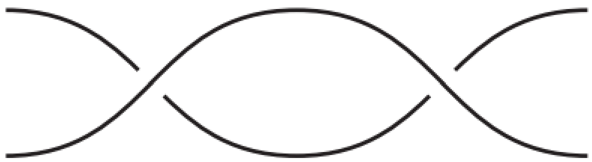

Figure 5: Unsimplified identity braid in B_2.

condition imposes that all braids are continuous. We also note that by the third condition braids cannot backtrack and must move monotonically along the figure. We show this further in Fig. 3.

Next, we introduce the identity of a braid group. Pictured in Fig. 4 is the lovely identity of B_2 (the braid group of two strings.)

Fantastic, but what if we told you that this braid is equivalent to the one in Fig. 5? That seems to be a bit of an issue. However, we can classify two braids as equivalent when we can move one braid, keeping it between the barriers of 0 and 1 and keeping the endpoints fixed to look like the other. For instance, in Fig. 5 we would move the strand with the lower two endpoints downwards while pushing the other strand upwards to form two parallel lines. We must guarantee that the endpoints will stay fixed. Otherwise, all braids are just the identity of their respective groups.

Now that we have the identity and composition figured out, it is a good time to move on to the inverses of braids. Well, we can simply find inverses by adding together braids which will cancel the identity as shown in Fig. 5 above. In essence, the inverse of a braid consists of that braid that undoes whatever the first braid did, which is obtained by flipping a braid representation across a vertical line going through its center. An example is following in Fig. 6. Notice that if the monotonic requirement were not true, we would be able to have knots in our strings which would lead to non-invertible strings.

Now that we have described some of the basics, we can go into a bit more group theory to comfort our minds. First off, a few rules about drawing braids. You can always use equivalence, defined above, to avoid having three strings cross at the same point. Furthermore, it is possible to avoid having two crossings happen directly above one another. The key to preventing this issue is to locate all crossings and shift half of them to the left and the other half to the right. A demonstration of poorly drawn braids and a correctly drawn braid is shown in Figs. 7 and 8, respectively. These two practices are standard and essential to the many times in your life when you will need to draw braid. Next, we make a slight

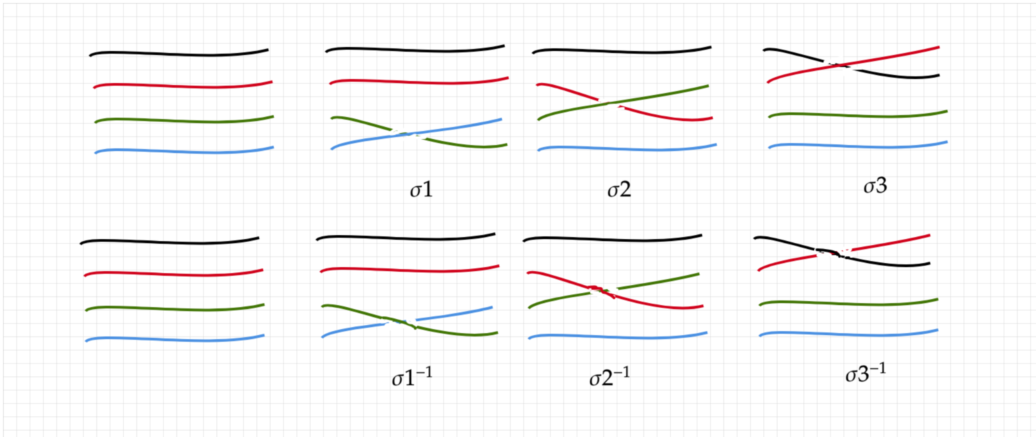

Figure 6: A set of 3 braids $\sigma_1, \sigma_2, \sigma_3$ and their respective inverses accompanied by the identity in the group B_4.

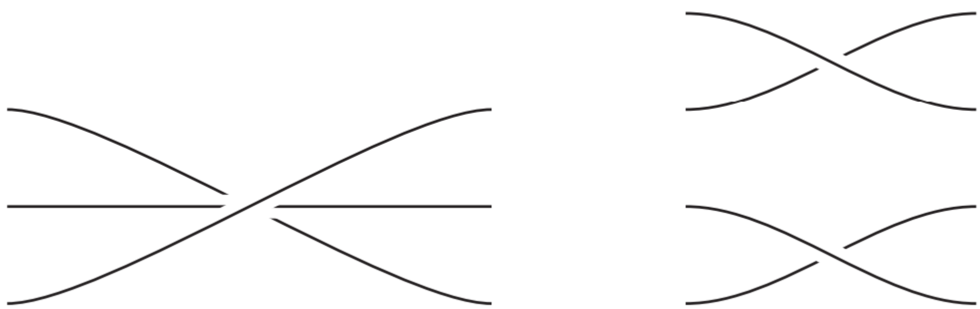

Figure 7: Two incorrectly drawn braids. On the left, a double-crossing occurs. On the right, two crossings occur one above the other.

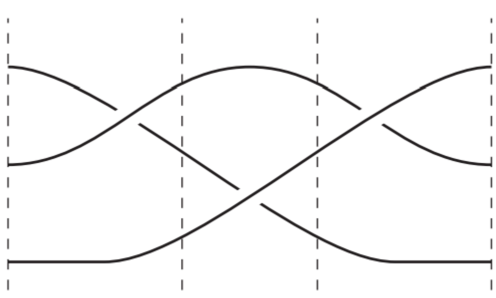

Figure 8: A correctly drawn braid is made by avoiding crossings that occur one above the other. The dashed lines are there to display the shift in string crossing to either direction.

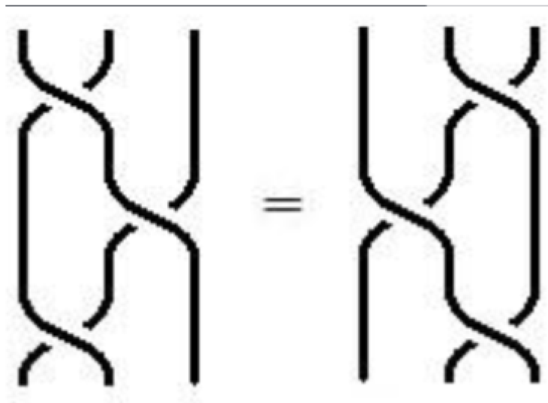

Figure 9: A representation of the relation in equation 1.

advancement in notation to make some of the next few theorems make sense. We say σ_i is the move that takes the i-th string over the (i+1)-th string. Thus all σ_i's serve as generators for our group and there are $n-1$ generators in a braid group of rank n, B_n. We can now consider the first question everyone asks about a well-defined group. Is it abelian? Unfortunately, the answer is no. Maybe this is fortunate since I'm pretty sure braid groups would be pretty boring if strings were abelian — specifically, all end points would be free to move and we would end up with a bunch of trivial groups. What is incredibly delightful, however, is the following relation that we can generally say

$$\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}. \tag{1}$$

This is one of the two relations which is needed to construct a braid group and is demonstrated in 9. The other is the commutativity of non-adjacent strings. Explicitly, $\sigma_i\sigma_j = \sigma_j\sigma_i$ if $|i-j| \neq 1$. These are known as the *braid relations.* For more intuition, Fig. 10 will help understand the second relation. From these two relations and their respective figures, we can also take away what compositions of σ generators may represent. Each composition is a crossing of strings. This will be helpful when we will deal with the elements of the braid group.

We can now formally present our braid presentation:

$$B_n = \langle \sigma_1, \sigma_2, \ldots \sigma_{n-1} || i-j| \neq 1, \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} \rangle, \tag{2}$$

for $i, j \leq n-1$. With this relation, we can state the following lemma which we will prove. It is a brief argument, however, it will elucidate the topics to come.

Lemma 1.2. *If $s_1, \ldots, s_{n-1}$ are elements of a group G satisfying the braid relations, then there is a unique group homomorphism $f : B_n \mapsto G$ such that $s_i = f(\sigma_i)$ for all $i = 1, 2, ..., n-1$.*

We will prove this directly, as we would for any old homomorphism. However, we will use our knowledge about the free group to make the ideas more concrete.

Proof: Let F_n be the free group generated by the set $\{\sigma_1, \sigma_2, \ldots \sigma_{n-1}\}$. There exists a unique homomorphism $\phi : F_n \mapsto G$ such that $\phi(\sigma_i) = s_i$ for all i. From this homomorphism,

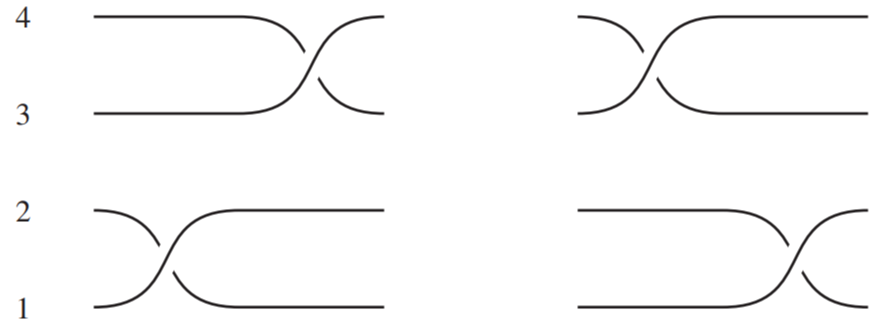

Figure 10: A demonstration of $\sigma_i\sigma_j = \sigma_j\sigma_i$ if $|i-j| \neq 1$. Explicitly, $\sigma_1\sigma_3 = \sigma_3\sigma_1$.

we can induce a group homomorphism $f : B_n \mapsto G$, provided $\phi(r^{-1}r') = 1$, or equivalently, $\phi(r) = \phi(r')$ for all braid relations $r = r'$. For the first braid relation, we get

$$\phi(\sigma_i\sigma_{i+1}\sigma_i) = \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} = \phi(\sigma_{i+1}\sigma_i\sigma_{i+1}) \tag{3}$$

and similarly for the second

$$\phi(\sigma_i\sigma_j) = \phi(\sigma_i)\phi(\sigma_j) = s_is_j = s_js_i = \phi(\sigma_i)\phi(\sigma_j) = \phi(\sigma_i\sigma_j). \tag{4}$$

These both satisfy the needs of a group homomorphism and thus we are done. □

We can now use this wonderful lemma to find out a bit more between the braid groups and another familiar family, the symmetric groups. Explicitly, consider the set of transpositions $s_1, \ldots, s_{n-1} \in S_n$. It is an exercise (left to the reader) to check that the transpositions satisfy the braid relations. Sure enough, there exists a unique group homomorphism $\pi : B_n \mapsto S_n$ such that $\pi(\sigma_i) = s_i$ for all $1 \leq i \leq n-1$. Moreover, the homomorphism is surjective since the set of transpositions generates S_n. A key corollary of this result is a proof that B_n with $n \geq 3$ is not abelian.

Proof: We know that the group S_n for $n \geq 3$ is nonabelian. Since there exists a surjective homomorphism from B_n to S_n, it follows directly that B_n must be nonabelian for $n \geq 3$. □

We now briefly discuss a few more important aspects of braid groups. First off, we define the pure braid group P_n.

Definition 1.3. The *pure braid group* P_n is defined as the kernel of the projection $\pi : B_n \mapsto S_n$, or in other words,

$$P_n = \ker(\pi : B_n \mapsto S_n).$$

The strings within pure braids are called *pure strings.* In a pure braid, pictured in Fig. 11, the beginning and the end of each strand are in the same position. Pure braid groups have more intrigue than might at first be noticeable. Pure braid groups are iterated semi-direct products of free groups. We will not prove this claim, however, we encourage the reader to discover more about it for themselves — see here [3].

Another important characteristic of braid groups is the abelianization of B_n which is defined by $B_n/[B_n : B_n]$. The most interesting part about examining the abelianization is the following theorem.

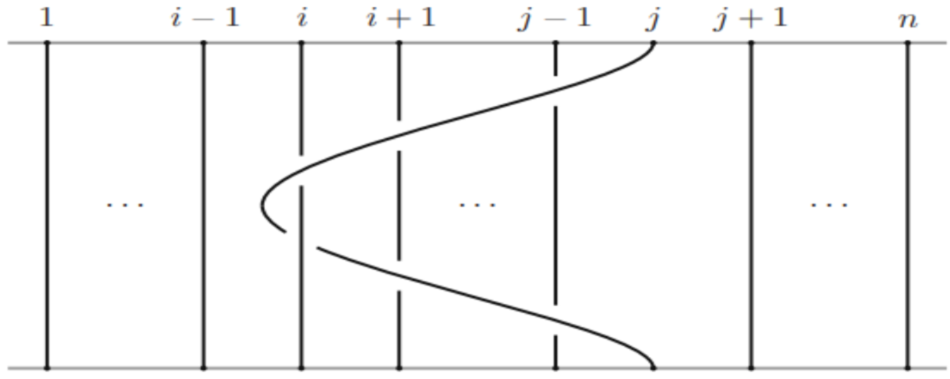

Figure 11: An example of a pure braid group.

Theorem 1.4. *The abelianization of B_n for $n \geq 2$ is an infinite cyclic group and therefore is isomorphic to $\mathbb{Z}$.*

We provide a proof to accompany the theorem. This takes the form of many proofs from class which we did on the homework concerning commutator subgroups.

Proof: First off, it is possible to show that the generators of B_n are all conjugates to one another, stemming from the first braid relation in equation 9. It follows that all generators of B_n have the same image on $B_n/[B_n : B_n]$. This implies that $B_n/[B_n : B_n]$ is cyclic since all generators of B_n will generate the entire group. Conversely, we define a mapping $\phi : B_n/[B_n : B_n] \mapsto \mathbb{Z}$ mapping each generator σ_i to the identity in $\mathbb{Z}$ and thus inducing a surjective homomorphism. It follows that since ϕ is surjective, then $B_n/[B_n : B_n]$ is infinite since $\mathbb{Z}$ is infinite. □

The final topic we will cover in detail is the center, which is equivalent to the center of pure braid groups. We already computed the center when we discussed abelianization. Namely, the center of B_n is $B_n/[B_n : B_n]$. However, there are a few more key points about the center we would like to make. The first is given in the following theorem.

Theorem 1.5. *If $n \geq 3$ then the center of B_n is equal to the center of P_n and is an infinite cyclic group generated by $\Theta_n = \Delta_n^2$, where*

$$\Delta_n^2 = (\sigma_1\sigma_2 \dots \sigma_{n-1})(\sigma_1\sigma_2 \dots \sigma_{n-2}) \dots (\sigma_1\sigma_2)\sigma_1.$$

This will not be proved now, however, in Fig. 12 we can see a representation of Δ^2. The beautiful corollary that follows is worth mentioning.

Corollary 1.6. *We say that for two integers m, n, if $m \neq n$, then the groups B_m and B_n are not isomorphic.*

What a comforting statement! We give a brief proof below, relying on the previous theorem.

Proof: We find from Theorem 1.4 that the image of $Z(B_n)$ in $B_n/[B_n : B_n] \cong \mathbb{Z}$ is a subgroup of $\mathbb{Z}$ with index $n(n-1)$. To show this we recall the generator of the center of B_n, Δ^2. Our mapping sends an element in the braid group to the sum of the powers on the

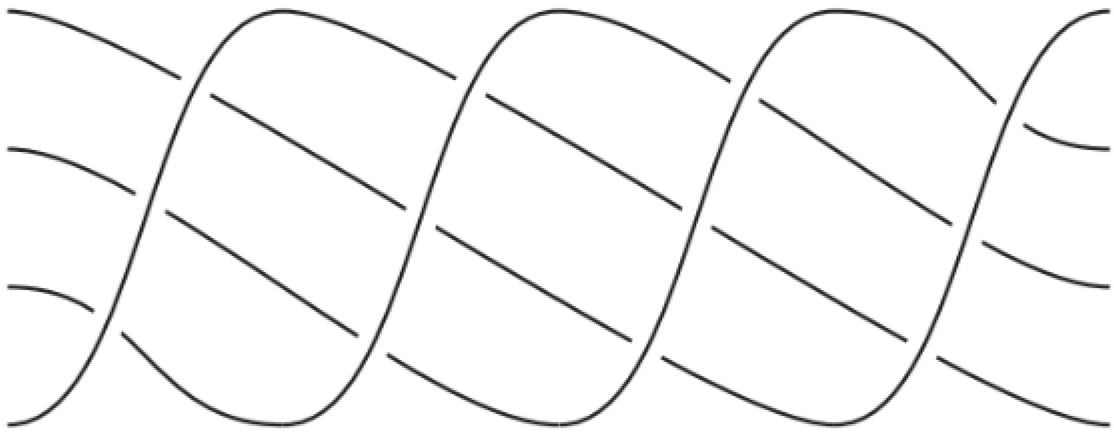

Figure 12: The full twist braid denoted as Δ^2.

generators, and since there are $n(n-1)$ generators in Δ^2 we assert that this is the index. It follows that if B_m is isomorphic to B_n, $m(m-1) = n(n-1)$ which implies $m = n$. □

With this simple and wonderful result, we conclude our discussion on braid groups for this blog post. As food for thought, we provide several more facts about braid groups which are thrilling to consider when one has more time.

Theorem 1.7. *Braid groups have a solvable word problem.*

An algorithm that works for the solution relies on the properties of pure braid groups. If a braid is not pure then it is most certainly not the identity. However, if it is a pure braid we must comb it to determine whether the braid will reduce. More on combing can be found here [2]. Unfortunately, the algorithm has a dreadful exponential time classification which means that when we have a braid group of rank n, the algorithm will take e^n steps to solve. This issue has been avoided and a quadratic classification solution — an exercise left to the reader!

Theorem 1.8. *Braid groups are torsion-free.*

This is another thrilling proof that requires further classifications of braid groups such as right and left-leaning braid groups [2]. Besides these theorems, there are also many connections to knot theory in mathematics where many theorems correspond between the two subjects.

Real-world applications like robotics and particle physics rely on the ability of braid groups to define three-dimensional space straightforwardly. Specifically, for particle spin, we can observe a particle-like object spinning in space and raveling a string over itself many times. However, in three dimensions, a particle will only ravel the string once and subsequently unravel it backward since the added dimension allow the string to go above and below the previously wound string. You may notice that braids are made in the same style and thus can only go above and below one another. In robotics, a similar idea is referred to as the configuration space [2] while in particle physics, this can be used as a proof to show that a particle can be either spin up or down with no other possibilities.

With this, I will bring this blog post to an end. Hope you now love braid groups as much as I do, and have a lovely break!

2 Acknowledgements

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References

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