Automaticity of Hyperbolic Groups

Sam Hiken

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1 Introduction

The last several weeks of our course on geometric group theory have covered two broad classes of groups: automatic groups and hyperbolic groups. An automatic group is one that permits an automatic structure, a set of finite automata that, taken together, can be used to solve Dehn's word problem on the group. A hyperbolic group is one that acts geometrically on a δ-hyperbolic space for some finite δ. While some may find it surprising, it turns out that one class of groups contains the other: any automatic group is in fact hyperbolic.

Recall the following definitions from class:

Definition 1.0.1. *An automatic structure for a group $G = \langle A|R\rangle$ is a set consisting of the following finite state automata:*

1. *W_{acc}: the word acceptor. $L(W_{acc})$ is a language such that each $g \in G$ is equal to at least one word in $L(W_{acc})$.*
2. *For each $a \in S \cup \{e\}$, W_a: the* multiplier *with respect to a, which accepts the language $\{(u, ua)|u \in L\}$.*

A group that permits such an automatic structure is said to be *automatic*. Automatic groups are really cool, in part because they come with a built-iin algorithm for solving Dehn's word problem. One natural question to ask is: which groups are automatic? The answer, it turns out, is *a lot*.

Definition 1.0.2. *For $\delta \geq 0$, a δ-hyperbolic space is a metric space where every triangle (that is, set of three points together with the geodesics between them) is δ-slim.*

A group is said to be *hyperbolic* if it acts geometrically on a δ-hyperbolic space. For our purposes, we will use the following equivalent definition:

Definition 1.0.3. *A group $G = \langle S|R\rangle$ is hyperbolic if (and only if) the Cayley graph $\Gamma_G(S)$ is δ-hyperbolic for some finite δ.*

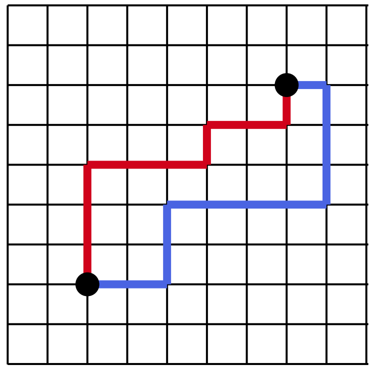
This is a very useful characterization of hyperbolic groups. The purpose of a Cayley graph is to geometrically capture group properties; true to form, the hyperbolicity of a group is captured in a property of its Cayley graph.

2 The k-fellow traveller property

In this section, we're going to introduce a property of graph paths called the *k-fellow traveller property*. First, let's introduce some notation. For a word $u \in S^*$ and a non-negative integer t, let $u(t)$ denote the first t letters of u if $t \leq |u|$, and let $u(t)$ denote u if $t \geq |u|$. If S is the generating set for a group G with Cayley graph $\Gamma_{G,S}$, let $\overline{u}$ denote the vertex in $\Gamma_{G,S}$ reached by following the path representing u starting at the identity vertex.

Definition 2.0.1. *Given a group G with Cayley graph $\Gamma_{G,S}$, two paths in $\Gamma_{G,S}$ obey the* k-fellow traveller property *if $d_{\Gamma_{G,S}}(\overline{u(t)}, \overline{v(t)}) \leq k$ for all $t \geq 0$.*

For instance, consider the following paths in the standard Cayley graph for $\mathbb{Z} \times \mathbb{Z}$:



These paths are 4-fellow travellers. Note that as we follow along the two paths, they are never of distance greater than 4 apart.

Here is a super neat lemma:

Lemma 2.1. *A group $G = \langle A|R \rangle$ is automatic if it satisfies the following two conditions:*

1. *G has a word acceptor W_{acc} that obeys condition 1 of the definition of automatic group.*
2. *There exists $k \in \mathbb{Z}^+$ such that if words $u, v \in L(W_{acc})$ where $d_{\Gamma_{G,S}}(\overline{u}, \overline{v}) = 1$, then u and v satisfy the k-fellow traveller property.*

Proof. To show G is automatic, we must prove the existence of a word acceptor, as well as a multiplier for each element of $A \cup \{e\}$. Luckily, condition 1 of our lemma allows us to assume the existence of a word acceptor W_{acc}. Moreover, we can assume W_{acc} is complete and deterministic, so each state has exactly one out-edge labeled with each generator. Below, we show the existence of a multiplier for each $a \in A \cup \{e\}$.

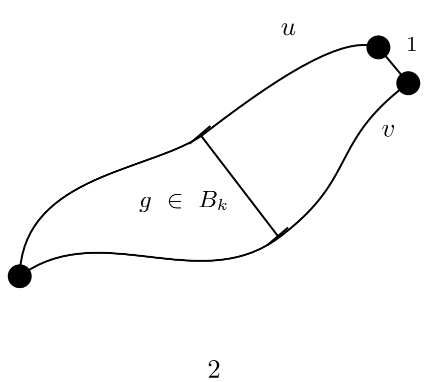
We'll build the multiplier from an FSA we'll call *Diff*. Let T denote the set of states in W_{acc}. Let B_k denote the set of vertices of distance at most k from e in $\Gamma_{G,A}$; that is, B_k is the k-ball around the identity. We'll define the state set for *Diff* to be $T \times T \times B_k$. Additionally, add a fail state s_{FAIL} to the state set of *Diff*. If t_0 is the start state of W_{acc}, then let the start state of *Diff* be given by (s_0, s_0, e).

Suppose *Diff* is in state (s_1, s_2, g). Let $(x, y) \in A \times A$. Let t_1 denote the state of W_{acc} reached by the out-edge of s_1 labeled with x. Let t_2 denote the state of W_{acc} reached by the out-edge of s_2 labeled with y. If $x^{-1}gy \notin B_k$, draw an edge from (s_1, s_2, g) to s_{FAIL} labeled with (x, y). Otherwise, draw an edge from (s_1, s_2, g) to $(t_1, t_2, x^{-1}gy)$ labeled with (x, y).

To construct $W_=$, mark any state of the form (s_1, s_2, e) as an accept state. To construct W_a for $a \in A \cup \{e\}$, mark any state of the form (s_1, s_2, a) as an accept state.

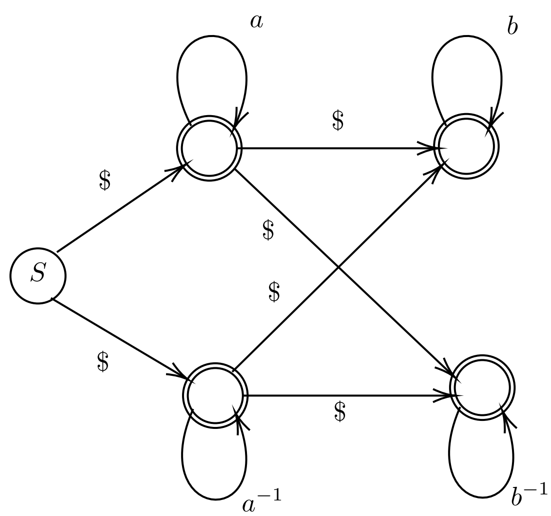
Now we show that $(u, v) \in L(W_a)$ if and only if $(u, v) \in L_a$. Let $u = x_1 x_2 ... x_m$ and $v = y_1 y_2 ... y_n$. If $(u, v) \in L(W_a)$, then $u^{-1}v = x_m^{-1} ... x_1^{-1} e y_1^{-1} ... y_n^{-1} = a$, so $(u, v) \in L_a$. The converse is also true; if $(u, v) \in L_a$ for some $a \in A \cup \{e\}$, then u and v are distance at most 1 apart in $\Gamma_{G,A}$. Because they obey the k-fellow traveller property, $\overline{u(t)}$ and $\overline{v(t)}$ are within distance k of each other in $\Gamma_{G,A}$ for any t, so the element $u(t)^{-1}v(t)$ is in B_k. Therefore, W_{acc} never goes to s_{FAIL} before reaching an accept state.

□

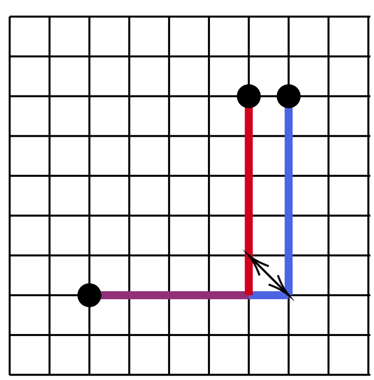


This lemma will be very useful for us, as it means we can prove a group is automatic without explicitly constructing every automaton in the group's automatic structure. Rather, we can construct the word acceptor, then prove a property of the group's Cayley graph.

Here's an example of the lemma in action: $\mathbb{Z} \times \mathbb{Z} = \langle a, b | ab = ba \rangle$. Consider the normal form for $\mathbb{Z} \times \mathbb{Z}$ given by $\{a^n b^m | n, m \in \mathbb{Z}\}$. We know that there exists a word acceptor W_acc that accepts this language.



Moreover, any two words that represent elements that are distance at most one apart in the group are also 2-fellow travellers in the standard Cayley graph for $\mathbb{Z} \times \mathbb{Z}$:



And, what do you know – $\mathbb{Z} \times \mathbb{Z}$ is automatic!

Interestingly enough, the converse of this theorem is actually true, as well. This means that we have a way of characterizing automatic groups as those groups who permit a word acceptor that accepts a language with a very particular geometric property. How fascinating!

3 Automaticity of hyperbolic groups

In this section, we will prove the main theorem of this project:

Theorem 3.1. *Any hyperbolic group is automatic.*

The proof proceeds in two parts:

1. Constructing the word acceptor. In particular, we'll have this word acceptor accept the language of geodesics in the Cayley graph of our group.

2. Proving that geodesics that represent group elements of distance at most 1 from each other obey the k-fellow traveller property.

We will actually use the k-fellow traveller property to construct the word acceptor, so we'll prove the second claim first.

3.1 k-fellow traveller property

Suppose u and v are geodesics that represent words accepted by W_{acc} such that $\overline{u}$ and $\overline{v}$ have distance at most 1 in $\Gamma_{G,A}$. Then the points $\overline{u}$, $\overline{v}$, and e form a triangle with sides u, v, and the edge $\langle u, v\rangle$ By Definition 1.0.1, this triangle is δ-slim for some finite δ. We'll show that, for any t,

$$d(\overline{u(t)}, \overline{v(t)}) \leq 2\delta + 1$$

implying u and t are $(2\delta + 1)$-fellow travellers.

Because our triangle is δ-slim, $\overline{u(t)}$ is of distance at most δ from some point x where either $x \in v$ or $x = \overline{u}$. In the latter case, x is adjacent to $\overline{v}$, so $\overline{u(t)}$ is of distance at most $\delta + 1$ from some point $x' \in v$. Then, $d(x', \overline{v(t)}) \leq \delta$. Else, v is not a geodesic, as the path along v to x' is not a geodesic. Therefore, $d(\overline{u(t)}, \overline{v(t)}) \leq \delta + \delta + 1 = 2\delta + 1$, proving the condition.

3.2 The word acceptor

Let $G = \langle A|R\rangle$ be a group with Cayley graph $\Gamma_{G,A}$. We'll show there exists an FSA W_{acc} that accepts the language of geodesics in $\Gamma_{G,A}$. Note that this provides us with a word acceptor for G, as any element in G can be expressed as a shortest path in $\Gamma_{G,A}$.

We'll let the state set of W_{acc} be the power set of B_k, together with a fail state s_{FAIL}, with out-degree zero. Mark $\{e\}$ as the start state, and every state except for s_{FAIL} as an accept state. Suppose the state T is not the fail state. Let g be a generator. If $g \in T$, draw an edge labeled g from T to s_{FAIL}. If $g \notin T$, draw an edge labeled g from T to the state:

$$\{g^{-1}ta | t \in T, a \in A \cup \{e\}\} \cap B_k$$

Suppose that, after reading in the word $u(\ell)$, our FSA is in the state T. We'll show that T consists exactly of the set of elements $x \in B_k$ such that $d(\overline{u(\ell)x}) = \ell$. We'll proceed by induction on ℓ:

Base case: $\ell = 0$. Then $u(\ell) =_G e$. For every $g \in A$, $g \notin T$ and $\overline{gu(\ell)} = g$ is a geodesic.

Inductive step: Suppose the statement holds for ℓ. Therefore, after reading in $u(\ell)$, the FSA is in the state $\{x | x \in B_k, d(\overline{u(\ell)x}, e) = \ell\}$. Then, after the FSA reads in g, it goes to the state

$$\{g^{-1}xa | x \in B_k, d(\overline{u(t)x}, e) = \ell, a \in A \cup \{e\}\} \cap B_k$$

Note that $\overline{u(\ell + 1)}g^{-1}xa = \overline{u(\ell)}xa$, which by the inductive hypothesis is of distance $\ell + 1$ from $\{e\}$. This proves the inductive step.

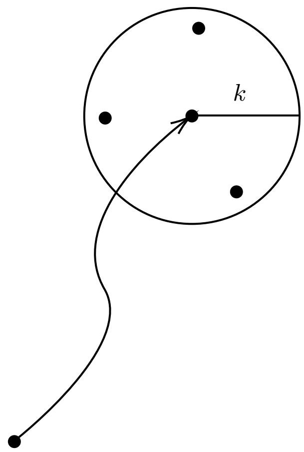


Figure: the FSA keeps track of all points within the k-ball around $\overline{u(\ell)}$ of distance ℓ from the origin.

Thus, the FSA goes to the fail state if and only if it reads in some generator g such that $d(\overline{u(\ell)g}, e) = \ell$.

We claim that our FSA accepts u if and only if u is a geodesic. Suppose u is a geodesic. Then there is no ℓ such that $d(\overline{u(\ell+1)}, e) = \ell$, so the FSA will never go to the fail state. Therefore, the FSA accepts any geodesic. To show the opposite direction, suppose u is not a geodesic. Let ℓ be the lowest value such that $u(\ell)$ is not a geodesic. Then $\ell = d(\overline{u(\ell)}) + 1$. To see why, note that if ℓ is lower than this value, $u(\ell)$ is a geodesic; if ℓ is greater than this value, $u(\ell - 1)$ is not a geodesic. Note that $\overline{u(\ell)}$ must be adjacent to the endpoint of a geodesic x, so $u(\ell)$ and x must be k-fellow travellers. Thus, on the ℓ-th step, the FSA moves to the fail state.

4 Concluding remarks

We have proven that any hyperbolic group permits an automatic structure. This is really cool for a couple of reasons. For one, it means that any property we can prove are possessed by automatic groups (like having solvable word problem) are also possessed by hyperbolic groups. Moreover, the theorem we have proven in this blog post means that automatic groups are a generalization of hyperbolic groups. Recall that a group is hyperbolic if and only if it permits a Cayley graph that is δ-slim for some δ. By Lemma 2.1, a group is automatic if and only if certain paths in its Cayley graph are k-fellow travellers, which is a distinct but similar condition.

5 References

Farb, B. Automatic Groups: a Guided Tour. *L'Enseignement Mathematique, Vol. 38, No. 1* (1992).